Random energy model at complex temperatures

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The complete phase diagram of the random energy model is obtained for complex temperatures using the method proposed by Derrida. We find the density of zeroes for the statistical sum. Then the method is applied to the generalized random energy model. This allowed us to propose an analytical method for investigating zeroes of the statistical sum for finite-dimensional systems.

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Nowdays the random energy model (REM) [1] is one of the most widely used models of statistical mechanics. Besides the direct use in spin-glasses the model has a wide range of applications in diverse areas of modern theoretical physics and biophysics. Recently the model found several unexpected applications in theories of well-developed turbulence [2], and strings [3–5]. On the other hand, it is well-known that the model enables one to treat the fundamental problems of information theory in terms of statistical mechanics. In particular, the problems connected with information transmission through a noisy channel can be formulated and effectively solved in the language of REM [6,7]. Though REM has a simple thermodynamic structure, at the same time it contains all essential ingredients of phase transition [1,8,9]. For temperatures less than the critical T_c (to be obtained based on clear and semi-intuitive physical reasons) the system is frozen in the spin-glass phase with one level of replica symmetry breaking. Along with this rough thermodynamical phase transition, there are some ''mild,'' mesoscopic phase transitions in the domain $(\tilde{T},T_c)$, $\tilde{T}>T_c$ [1,9].

It is well known that one can investigate the structure and properties of phase transitions by studying analytical properties of thermodynamical quantities in a complex temperature plane or/and magnetic field [10–12]. Indeed, if a phase transition is associated with a singular behavior of thermodynamic potential in the thermodynamic limit (in our case it is the free energy or statistical sum), then one can obtain important physical information by the consideration of its analytical properties in the complex plane. The method was proposed by Yang and Lee [13] (for the case of complex magnetic fields), and Fisher [14] (for the case of complex temperatures). It has a large variety of different applications in statistical physics; furthermore, it is one of the most exact and powerful methods for estimating critical indices and constructing phase diagrams. In particular, the method is applied to an investigation of phase transitions in disordered or strongly frustrated systems, since occasionally all other methods prove not so illustrating or are hardly realizable. Recently, a dense domain of phase transitions has been obtained with the help of this method in a nondisordered but fully frustrated model [12].

The REM has been investigated in a complex field/temperature plane [15,16] and the aim of the present paper is to continue this program, and to derive the complete phase diagram for generalizations of REM. Furthermore, we propose a method for investigating analytical properties of thermodynamical quantities of finite-dimensional systems. There is another, more sound motivation for the present investigation—recently it was shown that REM at complex temperatures is closely connected with strings [4,5]. In disordered systems there is an averaging with the help of canonical Gibbs distribution at given temperature, and averaging via the frozen disorder. In REM and related models the energy levels are considered to be random quantities, and the distribution is derived from a specific ''microscopic'' Hamiltonian [1,2],

$$H=-\sum_{1\leq i_1<i_2\cdots<i_p\leq N} j_{i_1\cdot i_p}s_{i_1}\cdots s_{i_p}. \tag{1}$$

Here s_i are ± 1 spins, $j_{i_1\cdots i_p}$ are quenched couplings with the normal distribution and a proper scale

$$\rho(j_{i_1\cdots i_p})=\sqrt{\frac{2C_p^N}{N\pi}}\exp\{-2C_p^N j^2_{i_1\cdots i_p}/N\}, \tag{2}$$

where $C_p^N=N!/p!(N-p)!$. It is possible to prove that in the limit of large p all energy levels are independently distributed with the probability

$$P(E)=\sqrt{\frac{1}{N\pi}}\exp(-E^2/N). \tag{3}$$

For the case of diluted coupling one keeps only αN nonzero members in Eq. (1) while the summation is performed over the set of indices $(i_1\cdots i_p)$. Now we again have independent random energies with the distribution

$$\rho(E)=\frac{1}{2\pi}\int_{-i\infty}^{i\infty}dx e^{-Ek-N\psi(x)}, \tag{4}$$

where

$$\psi(x)=\alpha\ln chx. \tag{5}$$

Case (3) corresponds to the choice

$$\psi(x)=\frac{x^2}{4}. \tag{6}$$

Let us define a partition function and a free energy. The consideration of complex temperatures is instrumental in the investigation of analytical properties of thermodynamical quantities. In particular, the conventional definitions of statistical physics should be adapted to this purpose. The partition function is defined as usual,

$$Z=\sum_{\alpha} e^{-\beta E_{\alpha}}.$$

Further, in the thermodynamical limit the expressions like

$$\langle \ln|Z|\rangle, \quad \langle \ln|\mathrm{Re}\, Z|\rangle \tag{7}$$

are identical. Below we will refer to the quantity $\langle \ln|\mathrm{Re}\, Z|\rangle$ as the free energy, taking into account that for real temperatures it is simply related to the usual free energy,

$$\begin{aligned}\langle \ln|\mathrm{Re}\, Z|\rangle &\equiv \frac{1}{2}\langle \ln|\mathrm{Re}\, Z|^2\rangle \\ &= \frac{1}{2}\left[\int_0^\infty dt\frac{e^{-t}}{t} - \left\langle \int_0^\infty dt \frac{e^{-t|\mathrm{Re}\, Z|^2}}{t}\right\rangle\right] \\ &= \frac{1}{2}\int_0^\infty dt\, \ln t e^{-t} + \int_0^\infty \ln t d\langle e^{-t^2|\mathrm{Re}\, Z|^2}\rangle \\ &= \frac{\Gamma'(1)}{2} + \frac{1}{2}\int_0^\infty \ln t d e^{-\phi},\end{aligned} \tag{8}$$

where $\langle\ \rangle$ means the averaging over Eq. (4). In Eq. (8) we made a substitution $t \to t^2$. Further,

$$\langle e^{-t^2|\mathrm{Re}\, Z|^2}\rangle = e^{-\phi} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dz e^{-z^2/4} f(zt)^{2N} \tag{9}$$

and

$$f(A)=\frac{1}{2\pi}\int_{-i\infty}^{i\infty} dx \int_{-\infty}^{\infty} dE e^{N\psi(x)-xE+i(A/2)[\exp(\beta E)+\exp(\bar{\beta}E)]}. \tag{10}$$

The function $e^{-\phi}$ in Eq. (8) is like the step function. It is equal to 1 at $-\infty<\ln t<-u_0$, then to an exponential accuracy it is zero in the interval $-u_0<\ln t<\infty$. The derivative of the step function is the δ function. So, in the thermodynamic limit Eq. (8) gives (u_0 to be defined later), that

$$\langle \ln|\mathrm{Re}\, Z|\rangle = u_0. \tag{11}$$

Let us perform integration over E by introducing the function $g(x)$. We have

$$\begin{aligned} f(A) &= \frac{1}{2\pi}\int_{-i\infty}^{i\infty} dx e^{N\psi(x)} \\ &\quad\times \int_0^\infty dy y^{x-1} e^{(iA/2)(y^{1+i\beta_2/\beta_1}+y^{1-i\beta_2/\beta_1})} \\ &= \int_{-i\infty}^{i\infty} dx e^{\psi(x)-x\ln A} g(x,A),\end{aligned} \tag{12}$$

where

$$g(x,A)=\int_0^\infty dy y^{x-1} e^{iy\cos[\beta_2/\beta_1 \ln(y/A)]}. \tag{13}$$

In Eq. (10) the integral is taken along the line that passes the point $x=0$ on the right-hand side. As we are interested in the region of values $A \ll 1$, so we have a large parameter $|\ln A|$ in the exponent, we move the integration loop to the position of the saddle point. If the saddle point x belongs to the interval $(0,-1)$, then the pole at $x=0$ is the only singularity to be intersected. The residue of our function is equal to 1. So, we obtain

$$f(A)-1 \approx \exp[N\psi(x) - x\ln A], \tag{14}$$

where x is determined from the equation of the saddle point

$$N\psi'(x)=\ln A \tag{15}$$

and the preexponents were omitted. Besides Eq. (14) there are another two asymptotic forms for $f(A,\lambda)$, when the saddle point intersects the singularities at points $x=-1$ and $x=-2$. One cannot calculate directly the expression for $f(A)$. To calculate $f(A)$ let us consider the derivatives

$$\begin{aligned} f'_A &= \frac{i}{4\pi}\int_{-i\infty}^{i\infty} dx \int_{-\infty}^{\infty} dE e^{-Ex+N\psi(x)+(iA/2)(e^{\beta E}+e^{\bar{\beta}E})} \\ &\quad\times (e^{\beta E}+e^{\bar{\beta}E}) \\ &= \frac{i}{4\pi}\int_{-i\infty}^{i\infty} e^{\psi[\beta(x+1)]-x\ln A} g(x,A) dx \\ &\quad+ \frac{i}{4\pi}\int_{-i\infty}^{i\infty} e^{N\psi[\bar{\beta}(x+1)]-x\ln A} g(x,A) dx. \end{aligned} \tag{16}$$

To identify the saddle point one has to move the integration line to the left. Again the pole is intersected at point 0,

$$f'(A) \approx \frac{i}{2}(e^{N\psi(\beta)}+e^{N\psi(\bar{\beta})}), \tag{17}$$

$$f(A)-1 \approx \frac{iA}{2}(e^{N\psi(\beta)}+e^{N\psi[\bar{\beta}]}). \tag{18}$$

Let us consider now the third type of asymptotics. The consideration of the second-order derivative gives

$$\begin{aligned} \frac{d^2 f}{dA^2} &= -\frac{1}{8\pi}\int_{-i\infty}^{i\infty} dx \int_{-\infty}^{\infty} dE e^{-xE+N\psi(x)+(iA/2)(e^{\beta E}+e^{\bar{\beta}E})} \\ &\quad\times (e^{2\beta E}+e^{2\bar{\beta}E}+2e^{2\beta_1 E}) \\ &= -\frac{1}{4\pi}\int_{-i\infty}^{i\infty} e^{N\psi[(x+2\beta_1)]-x\ln A} g(x) dx \\ &\quad - \frac{1}{8\pi}\int_{-i\infty}^{i\infty} e^{N\psi(x+2\beta)-x\ln A} g(x) dx \\ &\quad - \frac{1}{8\pi}\int_{-i\infty}^{i\infty} e^{N\psi[(x+2\bar{\beta})]-x\ln A} g(x). \end{aligned} \tag{19}$$

During the shift, the integration line intersects the pole, and one has

$$\frac{d^2 f}{dA^2}\approx-\frac{e^{N\psi[(2\beta_1])}}{2}, \tag{20}$$

$$f-1\approx-\frac{A^2 e^{N\psi(2\beta_1)}}{4}. \tag{21}$$

We have three asymptotic forms for the function $f(A)$: for the spin-glass phase (14), for the paramagnetic phase (18), and for the Lee-Yang-Fisher phase (21). We have

$$e^{-\phi}=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2/4+zi2^N t[e^{\alpha N\psi(\beta)}+e^{\psi(\bar{\beta})}]}, \tag{22}$$

$$e^{-\phi}=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp[-z^2/4-2^N e^{N\psi(\beta x)-x\ln(zt)}],$$
$$N\psi'(\beta x)=\ln(zt), \tag{23}$$

$$e^{-\phi}=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-z^2/4-z^2 2^4 N t^2 e^{\alpha N\psi(2\beta_1)}). \tag{24}$$

One can see that in the thermodynamic limit these expressions look like step functions. Using Eq. (11) we derive an expression for free energy.

Let us first consider case (1). For the paramagnetic phase

$$\langle\ln|\mathrm{Re}\,Z|\rangle=N\ln 2+\frac{N(\beta_1^2-\beta_2^2)}{4}. \tag{25}$$

For the spin glass phase with some replica symmetry breaking

$$\langle\ln|\mathrm{Re}\,Z|\rangle=N\sqrt{\ln 2}\,\beta_1. \tag{26}$$

For the phase without any magnetization, but with some Lee-Yang-Fisher zeros for the partition sum

$$\langle\ln|\mathrm{Re}\,Z|\rangle=\frac{1}{2}N\ln 2+\frac{\beta_1^2}{2}N. \tag{27}$$

Let us consider now the diluted version of the model. We take Eq. (5) for function ψ. For the paramagnetic phase

$$\langle\ln|\mathrm{Re}\,z|\rangle=N\ln 2+\alpha N\,\mathrm{Re}\ln\cosh\beta. \tag{28}$$

For the Lee-Yang-Fisher phase

$$\langle\ln|\mathrm{Re}\,z|\rangle=\frac{N}{2}\ln 2+\frac{\alpha N}{2}\ln\cosh 2\beta_1. \tag{29}$$

For the spin glass phase

$$\langle\ln\mathrm{Re}|z|\rangle=\alpha N\beta_1\tanh\beta_c, \tag{30}$$

where for β_c we have

$$\alpha[\beta_c\tanh\beta_c-\ln\cosh\beta_c]=\ln 2. \tag{31}$$

One can find the boundary lines between two phases from the coincidence of two equations. The boundary between Lee-Yang-Fisher and spin glass phases is the line

$$\beta_1=\beta_0,\quad \beta_0<\beta_2<\infty, \tag{32}$$

where the multicriticity point (β_1,β_0) lies on the intersection of line (37) with the boundary of spin glass and Lee-Yang-Fisher phases,

$$\sin^2(\beta_2)=\frac{2^{1/\alpha}-1-\tanh^2\beta_1}{2^{1/\alpha}(1-\tanh^2\beta_1)}. \tag{33}$$

For the case of ferromagnetic couplings we have the ferromagnetic phase with the old expression with $\beta_1\neq 0$, $\beta_2=0$. Let us consider the density of partition zeros near the β_1 axes. For the density of zeros we have the following expression [17]:

$$\frac{1}{2\pi}\left(\frac{d^2}{d\beta_1^2}+\frac{d^2}{d\beta_1^2}\right)\langle\ln|Z|\rangle. \tag{34}$$

For the Lee-Yang-Fisher phase we have

$$\rho(\beta_1,\beta_2)=\frac{1}{\pi}\frac{\alpha}{\cosh(\beta_1)^2}. \tag{35}$$

Near the paramagnetic–spin glass transition line we obtain

$$\rho(\beta_1,\beta_2)=\frac{1}{\pi}\frac{\alpha}{\cosh(\beta_1)^2}\delta(\beta_1+\beta_2-\beta_c)(\beta_1-\beta_c). \tag{36}$$

Now consider the phase structure of the generalized random energy model (GREM) in the domain of complex T. The spin glass phase transition conditions are the known ones with β_1 instead of $1/T$. The conditions for phase transition to the third Lee-Yang-Fisher phase are more complicated. We have to consider an expression like

$$\left\langle\prod_{\alpha}\prod_{\beta_\alpha}\exp i\frac{A}{2}[e^{\lambda_\alpha E_\alpha+\lambda_\beta E_\beta}+e^{\bar{\lambda}_\alpha E_\alpha+\bar{\lambda}_\beta E_\beta}]\right\rangle. \tag{37}$$

Upon expanding the exponent (to get the Lee-Yang-Fisher phase), we found

$$\left\langle\prod_{\alpha}\prod_{\beta_\alpha}1-A^2/4[e^{(\lambda_\alpha+\bar{\lambda}_\alpha)E_\alpha+(\lambda_\beta+\bar{\lambda}_\beta)E_\beta}]\right\rangle. \tag{38}$$

It follows from this expression that if the Lee-Yang-Fisher phase takes place at some level of hierarchy, then no paramagnetic phase will occur at higher levels, so the only possible choice will be the spin glass phase. This statement is, perhaps, the main point of the present work.

Let us consider the diluted version of GREM [18] with an infinite hierarchy number M. We can consider the case of large M with smooth distribution of z_k (the number of couplings at level k) and N_k (the number of the kth level branches is 2^{N_k}). In this case we can introduce a continuous variable $v=k/M$ with the definition range [0,1], which labels the level of hierarchy, and we define the distributions

$$z_k \equiv dz = z dv, \quad N_k \equiv dN = n'(v) \mathrm{d}v, \quad dv = \frac{1}{M}, \quad (39)$$

where $n(v)$ is the given function (the entropy in bits s). The variable v $(0<v<1)$ parametrizes the level of the hierarchical tree and z is just a parameter (for our spin system z is a total number of couplings and the parameter v labels the level of hierarchy). Similarly to the case of diluted GREM at real T one finds

$$-\frac{\beta F}{N} = z[1-v_2(\beta)]\mathrm{Re}\ln\cosh\beta + n[v_2(\beta)]\ln 2 + n[v_2(\beta)]\ln 2/2 + z[v_2(\beta)-v_1(\beta)]/2\ln\cosh 2\beta_1 + z\beta_1\int_0^{v_1(\beta)} \mathrm{d}v_0 g\left(\frac{z}{n'(v_0)}\right), \quad (40)$$

where $v_2(\beta)$, $v_1(\beta)$ are determined from the equations

$$2z\,\mathrm{Re}\ln\cosh\beta + n'[v_2(\beta)]\ln 2 = z\ln\cosh 2\beta_1 z, \quad (41)$$

$$\mathrm{Re}\ln\cosh\beta + n'[v_1(\beta)]\ln 2 = \beta_1 z g[z/n'(v_1)],$$

and the function $g(x)$ from

$$\frac{1}{2}(1+g)\ln(1+g) + \frac{1}{2}(1-g)\ln(1-g)] = \frac{\ln 2}{x}. \quad (42)$$

One can regard Eq. (40) as a free energy for a chain of subsystems. Some part of subsystems $0<v<v_1$ is in the spin glass phase, the part $v_1<v<v_2$ in Lee-Yang-Fisher, and the rest $v_2<v<1$ is in the paramagnetic phase. For the case of Edwards-Anderson model placed on a d-dimensional hypercubic lattice

$$z = Nd, \quad E = -vNd, \quad n(v) = \frac{Ns(-vdN)}{\ln 2}. \quad (43)$$

Here E is the energy and $s(E)$ is the entropy as a function of energy for the corresponding ferromagnetic Ising model. At given $\tilde{\beta}_0$, v_0 is defined via the energy of the ferromagnetic model on the same lattice at temperature $1/\tilde{\beta}_0$,

$$v_0(\tilde{\beta}_0) = -\frac{E(\tilde{\beta}_0)}{Nd}. \quad (44)$$

Here one can find $\tilde{\beta}_0$ according to the conventional definition

$$\frac{\mathrm{d}s}{\mathrm{d}E} = \frac{1}{\tau} \equiv \tilde{\beta}. \quad (45)$$

We obtain for the free energy

$$-\frac{\beta F}{Nd} = [1-v_2(\beta)]\mathrm{Re}\ln\cosh\beta + s[v_2(\beta)] + [v_2(\beta) - v_1(\beta)]/2\ln\cosh 2\beta_1 + s[v_2(\beta)]/(2Nd) + \beta_1\int_0^{v_1(\beta)} \mathrm{d}v_0 f\left(\frac{\ln 2}{\tilde{\beta}(v_0)}\right). \quad (46)$$

One can determine v_1, v_2 from the condition of the extremum for Eq. (46).

The method was proposed for approximate calculations in statistical mechanics of a disordered model on finite lattices. It enables one also to identify Lee-Yang-Fisher zeroes. In case of real temperatures our generalization of REM gives 5% accuracy. In our case we hope to achieve the same accuracy of calculations.

The solution of generalized REM (GREM) at complex temperatures was successful owing to a simple observation (38) that on the hierarchy of generalized REM a new (Lee-Yang-Fisher) phase will exist only at levels between paramagnetic and spin glass phases.

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